

Sliding phases in U(1) symmetric systems – mirage of the renormalization group

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We analyse the proposal of sliding phases (SP) in layers hosting global U(1) symmetric variables with finite inter-layer Josephson coupling. Based on the Kosterlitz-Thouless renormalization group (RG) approach, such phases were predicted to exist in various layered (or 1D quantum coupled) systems. The key in the RG argument is treating the coupling as though the variables are non-compact. Large scale Monte Carlo simulations of a layered model, where the SP is supposed to exist, finds no indication of such a phase. Instead, 3D behavior is observed. This result is consistent with the asymptotically exact analytical solution. A generic argument against SP in translationally invariant systems with short range interactions is provided. We have also suggested an alternative model for the SP – adding long-range interactions to the inter-layer Josephson term.

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I. INTRODUCTION

The idea behind the SP was put forward by Efetov in 1979 in the context of layered superconductor with parallel magnetic field¹. It was suggested that the field can suppress the inter-layer Josephson coupling so that the low energy properties of this 3D system can be described as being essentially of 2D character. Later, in Ref.² it was shown that the frustration due to the magnetic field is not sufficient to fully suppress the coupling.

In the context of quantum 1D chains (equivalent to 2D classical layers by the virtue of the quantum to classical mapping) the possibility of the decoupling between chains has also been explored^{3,4}. The main argument for such a decoupling is based on the scaling dimensions of the Josephson coupling determined with respect to the Luttinger liquid parameter in each chain: if it is larger than 2, the coupling should become irrelevant⁴. These proposals have been criticized in Refs.^{5,6} where it was shown that the inter-chain tunneling is always relevant.

Further argumentation in favor of the SP and, actually, the name *Sliding Phases* have been proposed in Refs.^{7,8} where the inter-layer gradient couplings between classical XY variables in each layer have been considered in addition to the Josephson one. Such gradient terms can independently control the scaling dimensions of the Josephson coupling and of the vortex fugacity in each layer so that the first one can become irrelevant above some temperature T_d (of the *dimensional reduction*⁹) which is below the temperature of the Berizinskii-Kosterlitz-Thouless (BKT) transition in the layers. Thus, there is a range of temperatures where the SP are supposed to exist. This approach was also developed for the case of quantum 1D Luttinger liquids coupled by both the Josephson and the gradient terms^{10,11} which are the analog of the Andreev-Bashkin drag effect¹².

It is important to note that the proposal of SP is based on applying the RG logic to compact variables characterized by global U(1) symmetry. While these early suggestions were more of a purely academic interest, expanding capabilities of ultra-cold-atoms techniques in recent

years emphasize the importance of these suggestions especially in the context of possible new phases in composite lattices¹³ and in the presence of disorder¹⁴. In more general terms, the question is if it is possible to realize a phase transition from a low- to higher- dimensional behavior.

Here we will analyze a simplest classical XY system characterized by the gradient interactions and the Josephson coupling u . The gradient terms are chosen in such a way that the SP is supposed to exist in some range where the renormalized value u_r of u scales to zero as layers size L grows. We will present the results of the large scale Monte Carlo simulations of this system. Our analysis is based on the dual formulation of the model – in terms of the closed loops. The main result is that no SP state exists in such a system. Instead, the value of u_r is always finite. This behavior will be compared with the standard asymmetric XY layered model where no SP are expected to occur. We will also derive the analytical result for u_r in the asymptotic limit when the intra-layer stiffness is much larger than u . The numerical results have been found to be consistent with the analytical solutions for both models.

Our paper is organized as follows. In Sec.II we introduce the bilayer model and provide the RG solution for SP. Then, we construct the dual representation in Sec. IIB. Using the duality we have found the asymptotic analytical solution for the renormalized Josephson coupling u_r in Sec. IIC. The Monte Carlo simulations of the bilayer model are presented in Sec.IID. Then, in Sec.III we present the results on a stack of bilayers along the same lines as for the bilayer. Finally, in Sec. IV we discuss the implications of our analytical and numerical results and also provide an alternative model for the SP.

II. BILAYER MODEL OF SP

Here we introduce a model of two asymmetric parallel layers, each being a square lattice of linear size $L = 1, 2, 3, \dots$ (in terms of the inter-site shortest dis-

tance) characterized by two fields $\psi_1 = \exp(\phi_1)$ and $\psi_2 = \exp(\phi_2)$ on the layers $z = 1, 2$, respectively. The action can be written as

$$H = - \sum_{\langle ij \rangle} [t_1 \cos(\nabla_{ij} \phi_1 - A_{ij}) + t_2 \cos(\nabla_{ij} \phi_2 - g_2 A_{ij})] + \frac{1}{2g} A_{ij}^2 - \sum_i u \cos(\phi_2(i) - \phi_1(i)) \quad (1)$$

where $t_1, t_2 > 0, g > 0$ and g_2 are parameters; $\langle ij \rangle$ denotes summation over nearest neighbor sites within each layer; $\nabla_{ij} \phi_a \equiv \phi_a(i) - \phi_a(j)$; A_{ij} is a bond vector field (that is, $A_{ij} = -A_{ji}$) oriented along the bond $\langle ij \rangle$. It is introduced in order to generate the "current-current" interaction (cf.⁷⁻¹¹) consistent with the compact nature of the fields $\phi_{1,2}$. This action is to be used in the partition function

$$Z = \int DAD\phi_1 D\phi_2 \exp(-H) \quad (2)$$

where the temperature is absorbed into the parameters t_1, t_2, u, g . Our focus is on verifying the applicability of the RG analysis to the renormalization of the Josephson coupling u . Hence, we will not discuss physical origins of the variables and the parameters.

A. The RG solution for bilayer

In the approximation ignoring compact nature of the variables, the terms $-\cos(\nabla_{ij} \phi_1 - A_{ij})$ and $-\cos(\nabla_{ij} \phi_2 - g_2 A_{ij})$ are replaced by $(\nabla_{ij} \phi_1 - A_{ij})^2/2$ and $(\nabla_{ij} \phi_2 - g_2 A_{ij})^2/2$, respectively. Then, the gaussian integration over A_{ij} can be carried out explicitly in Eq.(2), so that (1) in terms of the remaining variables becomes

$$H_0 = \frac{1}{2} \sum_{\langle ij \rangle} K_{ab} \nabla_{ij} \phi_a \nabla_{ij} \phi_b - \sum_i u \cos(\phi_2 - \phi_1), \quad (3)$$

where the matrix K_{ab} , $a, b = 1, 2$ is related to the original parameters as

$$K_{11} = \frac{t_1(1 + gg_2^2 t_2)}{1 + g(t_1 + g_2^2 t_2)}, K_{22} = \frac{t_2(1 + gt_1)}{1 + g(t_1 + g_2^2 t_2)}, \quad (4)$$

$$K_{12} = -\frac{gg_2 t_1 t_2}{1 + g(t_1 + g_2^2 t_2)}.$$

[As a matter of taste, we will keep $g_2 < 0$ in order to have $K_{12} > 0$]. The stability of H_0 is guaranteed by

$$K_{11}K_{22} - K_{12}^2 = \frac{t_1 t_2}{1 + g(t_1 + g_2^2 t_2)} > 0. \quad (5)$$

The condition for SP can be obtained along the lines of the logic^{4,7-11} which ignores the compactness of $\phi_{1,2}$. Specifically, introducing the variables $\varphi = \phi_1 + \phi_2$ and

$\theta = \phi_2 - \phi_1$ and, then, integrating out φ , the resulting partition function becomes

$$Z_0 = \int D\theta e^{-H_\theta}, H_\theta = \int d^2x \left[\frac{K}{2} (\vec{\nabla} \theta)^2 - u \cos \theta \right], \quad (6)$$

where the notation

$$K = \frac{K_{11}K_{22} - K_{12}^2}{K_{11} + K_{22} + 2K_{12}} \quad (7)$$

is introduced and the long wave limit is considered – so that the summation along the layers is replaced by the integration $\int d^2x \dots$

As long as the compactness of θ is ignored, Eqs. (6) represent the standard Sine-Gordon model in 2D. The RG analysis predicts (see in, e.g.,¹⁵) that at $K < K_d = 1/(8\pi)$ the renormalization renders the Josephson coupling u irrelevant in the thermo-limit $L \rightarrow \infty$. More specifically, the renormalized u should flow to zero as $u_r \sim uL^b \rightarrow 0$, $b = 2(1 - K_d/K) < 0$. Such a behavior is supposed to occur together with the persistence of the algebraic order along the planes. This requirement imposes further restrictions on the values of K_{ab} .

Without loss of generality let's assume $K_{11} < K_{22}$ and introduce the notations: $T = 1/K_{11}$ as a measure of temperature, and $Y = K_{22}/K_{11} > 1$, $X = K_{12}/K_{11}$. Then, the condition $K < 1/(8\pi)$ for SP becomes

$$T > T_d = \frac{8\pi(Y - X^2)}{1 + Y + 2X}. \quad (8)$$

In order to guarantee the algebraic order in each layer no BKT transition should occur in the layers. In order to determine possible types of vortices responsible for the transition, we examine the form (3) by "reinstating" the compactness of the variables in the limit $u = 0$ (which is supposed to renormalize to zero). In the presence of the gradient coupling a vortex may have a composite structure^{16,17}: q_1 circulations in the component 1 may be bound to q_2 circulations in the component 2. Finding the condition for the proliferation of such composite vortices can be achieved along the line of Kosterlitz-Thouless argument developed for simple XY-model in 2D. The free energy of such a complex is

$$F_v = \pi[K_{11}q_1^2 + K_{22}q_2^2 + 2K_{12}q_1q_2] \ln L - 2 \ln L. \quad (9)$$

Then, $F_v \propto \pi[(q_1 + Xq_2)^2 + (Y - X^2)q_2^2] - 2T$, and the stability against the BKT transition is guaranteed by the positivity of F_v or

$$T < T_{(q_1, q_2)} = \frac{\pi}{2} [(q_1 + Xq_2)^2 + (Y - X^2)q_2^2], \quad (10)$$

where the minimization with respect to q_1, q_2 must be performed. Proliferation of simple vortices $q_1 = \pm 1$, $q_2 = 0$ corresponds to $T_{(1,0)} = \pi/2$. It is, however, not always a minimal value as long as $X \neq 0$. Let's also note that, since $Y > 1$ by definition, there are no solutions for $T_d < T_{(q_1, q_2)}$ if $X = 0$.

Let's look for a solution when X is integer, that is, $X = 1, 2, 3, \dots$. Then, keeping in mind the stability requirement (5), that is, $Y - X^2 = \delta > 0$, the minimal vortex corresponding to $q_1 = -Xq_2$, $q_2 = \pm 1$ has free energy lower than that of the simple vortex if $\delta < 1$. In this case, the solution for $T_d < T_{(X,-1)}$ exists if $X \geq 3$. Proliferation of the composite vortex pairs corresponds to disordering of the original fields $\exp(i\phi_{1,2})$, while the composite field $\Psi = \exp(i(\phi_1 + X\phi_2))$ remains (algebraically) ordered. This mechanism constitutes the formation of thermally induced bound phases (or using the language of superfluidity – *Thermally Paired Superfluid*, TPS,¹⁸). In other words, the system behaves in a such a way that the algebraic order persists only in the composite field $\Psi = \psi_1(\psi_2)^X$. [Since $X > 1$ we call such a composite phase as thermally bound superfluid (TBS) by analogy with the TPS]. This effect does not require that X is necessarily integer. If X is non-integer, its closest integer part will determine the power of ψ_2 .

If $\delta > 1$, the lowest energy vortex is the simple one, and the solution for the inequality $T_d < T_{(q_1,q_2)}$ also requires $X \geq 3$. For $X \gg 1$, $T_d \rightarrow 0$ while $T_{(q_1,q_2)} \rightarrow (\pi/2)\min(1, \delta)$ as long as δ is kept constant. Such a limit corresponds to the largest range of T where SP are to be anticipated for the two-layer model. However, for practical purposes of simulations using too large X leads to slower convergence. Thus, we choose $X = 5$, $Y = 25.5$ corresponding to a reasonably wide range where SP is anticipated to exist. Taking into account Eq.(7), the condition for the SP can be written as

$$\frac{8\pi\delta}{\delta + (1 + X)^2} < T < \frac{\pi}{2}\delta. \quad (11)$$

Keeping in mind the chosen values $X = 5, \delta = 1/2$, this becomes $8\pi/73 < T < \pi/4$ or $0.344 < T < 0.785$. The results of the simulations will be conducted at the "temperature" T in the middle of the interval (T_d, T_{BKT}) , that is, $T \approx 0.565$. More specifically, $K_{11} = 1/T$, $K_{22} = 25.5K_{11}$, $K_{12} = 5K_{11}$. In terms of the original parameters t_1, t_2, g_2, g , the relations $gt_2|g_2|(1 - 5|g_2|) = 5$, $gt_1(5.1|g_2| - 1) = 1$, $t_1 \approx 0.177|g_2|/[(1 - 4|g_2|)(5.1|g_2| - 1)]$ and $10/51 < |g_2| < 1/5$ must be satisfied. [Such a fine tuning is of no concern, since the focus is on the consistency of the paradigm of SP rather than on physical origin of the values].

Concluding this section, the presented analysis based on the RG finds that it is possible to find a range of temperatures where the sequence of phases is as presented in Fig. 1 in the panel (a): at $T < T_d$ the Josephson coupling is relevant. We call this range as Josephson phase in Fig. 1. At $T_d < T < T_{(q_1,q_2)}$ there is the SP where the symmetry $U(1)$ is promoted to $U(1) \times U(1)$. In the range $T_{(q_1,q_2)} < T < T_n$ the TBS phase is characterized by the composite field Ψ . Thus, the broken symmetry is partially restored through the subgroup Z_N , where $N = 1 + [X]$. At higher temperatures $T > T_n$ the composite field Ψ becomes disordered.

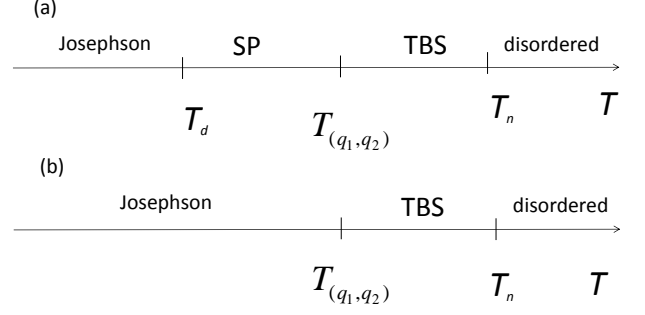


FIG. 1: (Color online) Two options for phases in the bilayer model: (a) with the SP according to RG; (b) Without SP.

B. Dual representation

The partition function Z can be evaluated by the high-temperature expansion method (see e.g. in¹⁹) in terms of t_1, t_2, u and the explicit integration over the variables. This approach allows obtaining Z in terms of the integer bond variables – powers of the corresponding Taylor series. Since the resulting configurational space consists of closed loops of the bond currents, further simulations can be effectively performed by the Worm Algorithm²⁰. As will be also shown, the language of loops also allows obtaining analytic expressions for the renormalized Josephson coupling u_r which are exact in the asymptotic limit.

We will be utilizing the Villain approximation²¹ for the cosines to obtain the so called J-current version²² of Eqs.(2),(1):

$$Z = \sum_{\{m_{a,ij}, m_i\}} \int D\phi \int DA e^{-H_V}, \quad (12)$$

$$\begin{aligned} H_V = & \sum_{\langle ij \rangle} \left[\frac{\tilde{t}_1}{2} (\nabla_{ij}\phi_1 - A_{ij} + 2\pi m_{1,ij})^2 \right. \\ & + \frac{\tilde{t}_2}{2} (\nabla_{ij}\phi_2 - g_2 A_{ij} + 2\pi m_{1,ij})^2 + \frac{1}{2g} A_{ij}^2 \Big] \\ & + \sum_i \frac{u_V}{2} (\phi_2(i) - \phi_1(i) + 2\pi m_i)^2, \end{aligned} \quad (13)$$

where $m_{a,ij} = -m_{a,ji} = 0, \pm 1, \pm 2, \dots$ ($a = 1, 2$) are integer numbers defined along bonds between two nearest sites i and j along the planes and $m_i = 0, \pm 1, \pm 2, \dots$ is an integer assigned to a site i and oriented from the layer 1 to the layer 2.

The Villain approximation proves to be very accurate for establishing the transition points as well as in general if the effective constants $\tilde{t}_1, \tilde{t}_2, u_V$ are properly expressed in terms of the corresponding bare values t_1, t_2, u (see in Ref.²³). The "renormalization" can be essentially ignored for $t_1, t_2 \geq 1$, so that in what follows we will be using $\tilde{t}_1 = t_1, \tilde{t}_2 = t_2$. Similarly, for the Josephson coupling

$u \sim 1$ one should take $u_V = u$ and, if $u \ll 1$, the corresponding relation is $u_V = 1/(2 \ln(2/u))^{21,23}$.

After using the Poisson identity for each integer and performing the integrations over ϕ_i and A , the resulting expression becomes

$$Z = \sum_{\{J_{1,ij}\}, \{J_{2,ij}\}, \{J_{z,i}\}} e^{-H_J}, \quad (14)$$

$$H_J = \sum_{\langle ij \rangle; a,b} \frac{1}{2} (K^{-1})_{ab} J_{a,ij} J_{b,ij} + \sum_i \frac{1}{2u_V} J_i^2, \quad (15)$$

where $a, b = 1, 2$ labels layers and $(K^{-1})_{ab}$ is the matrix inverse to K_{ab} introduced in Eqs.(3),(4); The summation runs over the integer bond currents $J_{1,ij}, J_{2,ij}$ (oriented from site i to site j so that $J_{a,ij} = -J_{a,ji}$, $a = 1, 2$) within each corresponding layer 1,2 as well as over the integer currents J_i oriented along the bond connecting the site i in the layer 1 to the site i in the layer 2. All the configurations are restricted by the Kirchhoff's current conservation rule – the total of all J-currents incoming to any site must be equal to the total of all outgoing currents from the same site.

It is useful to note that Eqs.(14,15) can also be obtained directly from Eq.(3) by reinstating the compact nature of the variables: $\nabla_{ij}\phi_1 \rightarrow \nabla_{ij}\phi_1 + 2\pi m_{1,ij}$ and $\nabla_{ij}\phi_2 \rightarrow \nabla_{ij}\phi_2 + 2\pi m_{2,ij}$ so that the matrix K_{ab} is viewed as being independent from the parameters in the action (1).

The system (14,15) features statistics of closed loops. If $u = 0$, there are two sorts of loops – one in each layer. Thus, each configuration is characterized by definite values of the windings $W_{a,\alpha}$ in the a th layer along the $\alpha = \hat{x}, \hat{y}$ directions of the planes. It is straightforward to show that statistics of these windings determine the renormalized values \tilde{K}_{ab} of the matrix K_{ab} along the line of the approach²⁴. More specifically

$$\tilde{K}_{ab} = \frac{1}{2} \sum_{\alpha=\hat{x},\hat{y}} \langle W_{a,\alpha} W_{b,\alpha} \rangle. \quad (16)$$

This expressions are valid for periodic boundary conditions (PBC). It is important to note that \tilde{K}_{ab} represents an exact linear response (at zero momentum) with respect to Thouless phase twists. In other words, if there are externally imposed infinitesimal constant gradients $\nabla_\alpha \phi_{1,2} \rightarrow 0$ (violating the PBC) of the phases $\phi_{1,2}$, the free energy acquires the contribution $\delta F = \frac{1}{2} L^2 \sum_{a,b,\alpha} \tilde{K}_{ab} \nabla_\alpha \phi_a \nabla_\alpha \phi_b$. On the other hand, in the presence of the gradient the integrand of the partition function gets the factor $\exp(iL \sum_{a,\alpha} W_{a,\alpha} \nabla_\alpha \phi_a)$. Comparing both expressions leads to the relation (16).

As a test of consistency, we have checked numerically that in the regime where the SP state is supposed to exist (that is, $X = 5, Y = 25.5, T \approx 0.565$), the deviations of \tilde{K}_{ab} from the bare values K_{ab} are within the statistical error less than 1% for all tested sizes of the layers

$10 \leq L \leq 1000$. Significant deviations are observed only as the system approaches fully disordered state – that is, where the fields $\psi_{1,2}$ as well as the composite one Ψ become disordered. In this case, \tilde{K}_{ab} flow to zero as L increases. The deviations remain small (about 2-3%) even in the regime where Ψ is the only ordered field. The emergence of the TBS is detected by observing that windings $W_{a,\alpha}$ in the layers 1 and 2 are changing exactly by the increment $\Delta W_1 = 1, \Delta W_2 = X$ (plus or minus), respectively.

At finite values of u the loops belong to both layers so that no separate windings can be introduced. However, the sums $W_\alpha = W_{1,\alpha} + W_{2,\alpha}$ remain well defined and can be used to evaluate the rigidity ρ_α of the fields along the layers. In a general case of N_z symmetric (with respect to the x, y directions) layers $\rho = \rho_x = \rho_y$:

$$\rho = \frac{1}{2N_z} \sum_\alpha \langle W_\alpha^2 \rangle \quad (17)$$

$$W_\alpha = \frac{1}{L} \sum_{\langle ij \rangle, a=1,2,\dots,N_z} J_{a,ij}, \quad (18)$$

where for a given $\alpha = \hat{x}, \hat{y}$ in (18) the bond $\langle ij \rangle$ (as well as $J_{a,ij}$) is oriented along the direction α .

Our focus here on the renormalized value u_r of the Josephson coupling u in the SP regime. If the periodic boundary conditions are also imposed perpendicular to the layers (along z -direction), the inter-layer response u_r is given by windings W_z along z -direction:

$$u_r = \frac{N_z}{L^2} \langle W_z^2 \rangle, \quad W_z = \frac{1}{N_z} \sum_i J_i, \quad (19)$$

where the summation \sum_i of the currents J_i (oriented along z -direction) is performed over all sites of all layers. Similarly to the cases (16) and (17), Eq.(19) represents the full linear response at zero momentum – that is, the renormalized value u_r of the Josephson coupling u .

At this point, we should comment on how to interpret the PBC for two layers, $N_z = 2$. While in the case $N_z \geq 3$ it is a natural procedure to link the $z = N_z$ th layer to the first one, $z = 1$, by the Josephson term, the case $N_z = 2$ needs an auxiliary construction because the layers 1 and 2 are coupled already directly. The formal procedure, then, consists of adding a third layer, $z = 3$, with no rigidity along x, y directions and coupled by the Josephson term to both layers, $z = 1, 2$. If the coupling u_{13} between the layers 1 and 3 and the coupling u_{23} between the layers 2 and 3 add up as $1/u_{13} + 1/u_{23} = 1/u_V$, in the dual action (15) the sum in the last term can be extended to the layers $z = 1, 2, 3$, while the first term is still confined to the layers $z = 1, 2$. The key to this procedure is the Kirchhoff's rule: the J-current from a site (x, y) along z -direction from the layer 2 to the layer 3 must be exactly the same as the current from the site (x, y) in the layer 3 to the layer 1. Then, in the form (15) the same value u_V can be used for the currents from the layer 1 to the layer 2 directly or through the layer 3.

C. Asymptotic expression for u_r

As mentioned above, the dual representation allows obtaining analytically the asymptotic values for u_r . Let's begin with the trivial case of zero stiffnesses K_{ab} and arbitrary number of layers, $N_z = 2, 3, 4, \dots$. In this case, the action (15) trivially becomes

$$H_A = \frac{N_z}{2u_V} \sum_i J_i^2, \quad J_i = 0, \pm 1, \pm 2, \dots \quad (20)$$

where the summation runs over all sites i of only *one* layer, say, $z = 1$. In this expression the Kirchhoff rule dictates that the current J_i at a given site along z -direction must be the same for all values of z . Thus, such a current with $J_i = \pm 1$ constitutes one closed loop characterized by the winding $W = J_i$. This allows constructing the partition function exactly as

$$Z_A = \left[\sum_{W=0, \pm 1, \pm 2, \dots} \exp \left(-\frac{N_z}{2u_V} W^2 \right) \right]^{L^2} \quad (21)$$

where L^2 is the number of sites in one layer. The stiffness (19) can be found by taking into account that the total winding along z -direction is $W_z = \sum_i J_i$, where the summation runs over L^2 sites of only one layer. Then, using statistical independence of different sites we find

$$u_r = \frac{2N_z \sum_{W=1,2,\dots} W^2 \exp(-N_z W^2/2u_V)}{1 + 2 \sum_{W=1,2,\dots} \exp(-N_z W^2/2u_V)}. \quad (22)$$

This expression shows that, as long as N_z is finite, the Josephson coupling remains relevant even if there is no in-plane order.

In the limit $u_V \ll 1$ only the term $W = 1$ is important, so that Eq.(22) becomes

$$u_r = \frac{2N_z}{2 + \exp(N_z/2u_V)}. \quad (23)$$

Eq.(23) can be used even in the case when there is finite stiffness along the layers, with N_z substituted by some effective value M , that is,

$$u_r = \frac{2M}{2 + \exp(M/2u_V)}. \quad (24)$$

This is easily justified in the "ideal gas" approximation of rare fluctuations of the J-currents in z -direction. In the case of $K_{12} = 0$ the asymptotic asymmetric limit corresponds to $K_{11} = K_{22} \gg 1$ (so that the system is well above the disordering transition) and $u_V \ll 1$. The loop proliferation can be viewed from the perspective of the Worm Algorithm²⁰ where one open end of a string of J-currents walks randomly until it meets another open end so that the loop is formed. Then, the most of the path is residing in a layer with only occasional jumps between neighboring layers. These jumps are controlled

by the exponential smallness of $\exp(-1/2u_V)$. Thus, the full action can be well approximated by H_A , Eq.(20), with $N_z = 1$. This leads to Eq.(24) with $M = 1$:

$$u_r = \frac{2}{2 + \exp(1/2u_V)}. \quad (25)$$

Later we will present the numerical evidence that the stiffness perpendicular to the layers of a simple XY layered model in the asymptotic limit can well be described by the above equation. Below we will show that for the case of the two asymmetric layers, the effective value of M is $M = 2$ in Eq.(24).

D. Numerical results for $N_z = 2$

Here we present the results of Monte Carlo simulations of the bilayer in the regime of SP in the limit $u_V \ll 1$. The action (15) can be represented in the notations T, X, Y, δ as

$$H_J = \sum_{\langle ij \rangle} \left[\frac{T}{2} J_{1,ij}^2 + \frac{T}{2\delta} (J_{2,ij} - X J_{1,ij})^2 \right] + \sum_i \frac{J_i^2}{2u_V} \quad (26)$$

where $J_{1,ij}$ and $J_{2,ij}$ refer to the inplane bond currents in the layers 1 and 2, respectively; the actual values of the parameters used in the simulations have been discussed at the end of Sec.II A: $X = 5, \delta = 1/2, T = (T_d + T_{(X,-1)})/2 \approx 0.565$.

The structure of the loops is determined by the energy of creating a J-current element along a given direction. A typical energy to create an additional J-current element in the plane 2 can be estimated as $\delta E_2 \approx T/(2\delta) \approx 0.5$. Thus, large loops with a typical values $|\vec{J}_2| = 1$ can exist in the plane 2. In contrast, the energy to create an isolated element in the plane 1 (with no J_2 currents along the same bond in the layer 2) costs much larger energy: $\delta E_1 \approx T(1 + X^2/\delta)/2 \approx 15$. Thus, the probability to create such an element is exponentially suppressed as $\sim \exp(-15)$, and no entropy contribution (due to 4 optional directions along the plane) can compensate for such a low value. This implies that no large isolated loops can exist in the layer 1. The only option to create a large loop in the layer 1 is if each element $J_{1,ij}$ is coupled to currents $J_{2,ij} = X J_{1,ij}$ along the same bonds in the layer 2. A typical energy of this combined element is $\delta E_{12} \approx T/2 \approx 0.25$. This strong asymmetry between the layers has immediate implication for the windings along z -direction – the minimal length M of the element J_i must be $M = 2$ in Eq.(24). Thus, the stiffness u_r in the limit $u \ll 1$ becomes

$$u_r = \frac{4}{2 + \exp(1/u_V)} \approx 4e^{-1/u_V} = u^2, \quad (27)$$

where the asymptotic expression $u_V = \frac{1}{2 \ln(2/u)}$ ^{21,23} has been used. The results of the simulations is shown in Fig.2. The first striking feature to notice is that u_r does

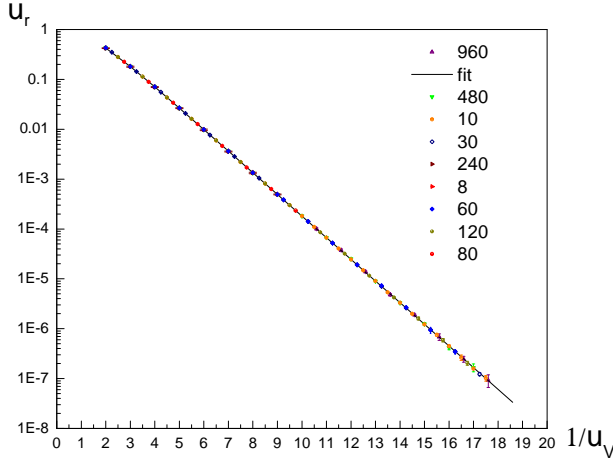


FIG. 2: (Color online) Monte Carlo results for the inter-layer stiffness u_r vs its bare value u_v of two layers for various layer sizes (shown in the legend). Error bars are shown, and for the majority of the data points these are smaller than symbols. The fit line is the solution (27).

not depend on the layers size L over 2 orders of magnitude of L and over 7 orders of u_v (which is actually $\sim \ln(1/u)$ of the bare coupling). Second, the numerically found value u_r follows the analytical result (27) with high accuracy – even for values $u_v \sim 1$. Both features are in the striking conflict with the RG prediction stating that u_r should decay as $\propto L^{2(1-T/T_d)} \approx L^{-1.28} \rightarrow 0$ in the SP regime ($T > T_d$).

It should be also noted that the stiffness along the layers (17) remains finite and much larger than u_r , that is, $\rho = 32.3 \pm 0.1$ for all simulated sizes from $L = 8$ to $L = 960$. This justifies the validity of Eq.(27) even in the case $u_v \sim 1$.

III. EXTENDING THE TWO-LAYER MODEL TO ARBITRARY NUMBER $N_z/2$ OF PAIRS OF LAYERS

As it became evident from the previous analysis, no SP can occur in the double layer. Referring to the sketch of the possibilities, Fig. 1, the option (b) is realized. Here we will address a possibility of SP in a N_z -layers setup. In other words, we will be looking for a behavior where the renormalized stiffness (the inter-layer Josephson coupling) u_r decays as a function of N_z in the limit $L \rightarrow \infty$, while the stiffness along planes remains finite. [This would be a "weaker" version of the SP].

A. RG solution

We consider the PBC setup: the odd $z = 1, 3, 5, 7, \dots$ and the even $z = 2, 4, 6, \dots$ layers are characterized by the inplane stiffnesses K_{11} and $K_{22} > K_{11}$, respectively, with the nearest layers coupled by the current-current term $\propto K_{12}$ (the same for all pairs of layers) as well as by

the Josephson coupling $-u \sum_{x,y,z} \cos(\phi_{z+1} - \phi_z)$, where $\psi_z(x, y) = \exp(i\phi_z(x, y))$ is the XY variable defined on a site (x, y) belonging to the layer z .

In the linearized with respect to $\nabla_{ij}\phi_z$ approximation analogous to Eq.(3) the model becomes

$$H_0 = \sum_{z=1,3,5,\dots} \left\{ \sum_{\langle ij \rangle} H_{z;ij} - u \sum_{x,y} [\cos(\phi_{z+1} - \phi_z) + \cos(\phi_{z-1} - \phi_z)] \right\} \quad (28)$$

where the summation runs over odd values of z and the notation

$$H_{z;ij} = \frac{K_{11}}{2} (\nabla_{ij}\phi_z)^2 + \frac{K_{22}}{2} (\nabla_{ij}\phi_{z+1})^2 + K_{12} \nabla_{ij}\phi_z (\nabla_{ij}\phi_{z+1} + \nabla_{ij}\phi_{z-1}) \quad (29)$$

is used.

If the compact nature of ϕ_z is ignored, the one-loop RG flow equation for u_r reads

$$\frac{du_r}{d \ln L} = \left(2 - \frac{1}{2S} \langle (\phi_{z+1} - \phi_z)^2 \rangle_S \right) u_r \quad (30)$$

where $\langle \dots \rangle_S$ refers to the RG shell integration over the inplane momenta $\Lambda/(1+S) < |q| < \Lambda$, with $\Lambda \sim 1/L$ being the cutoff and $S \rightarrow 0$. We note that, due to the PBC along z -direction, the mean $\langle (\phi_{z+1} - \phi_z)^2 \rangle$ does not depend on z .

Using discrete Fourier representation along z direction with doubled unit cell containing two layers (the odd and the even) with two sorts of phases $\phi_z = \phi^{(1)}(z)$ and $\phi_z = \phi^{(2)}(z)$ along odd and even layers, respectively, the part $H_{z;ij}$ can be diagonalized and the correlator in Eq.(30) found. This gives Eq.(30) rewritten as

$$\frac{du_r}{d \ln L} = 2 \left(1 - \frac{T}{T_d} \right) u_r, \quad (31)$$

where

$$T_d^{-1} = \frac{1}{4\pi N_z} \sum_{m=0}^{(N_z/2)-1} \frac{1 + Y + 4X \cos^2 q_m}{Y - 4X^2 \cos^2 q_m}, \quad (32)$$

and the wavevectors along z take values dictated by the periodic boundary conditions $q_m = 4\pi m/N_z$, $m = 0, 1, 2, \dots, (N_z/2) - 1$. Here we use the same notations $T = 1/K_{11}$, $X = K_{12}/K_{11}$, $Y = K_{22}/K_{11}$ introduced in Sec.II A. Thus, at $T > T_d$ RG predicts irrelevance of u_r .

The upper limit on T can be obtained from the requirement of no free 2D vortices in the limit $u = 0$. There are, actually, two options: i) looking for a composite vortex characterized by phase windings q_1 and q_2 in odd and even layers, respectively, forming a string of length N_z perpendicular to the layers; ii) considering independent vortices $q_1 = \pm 1$ only in odd layers (characterized by smallest stiffness K_{11}) and characterized by the temperature $T_{(1,0)} = \pi/2$. As the analysis shows, the option when

composite vortices form finite strings (say, $q_2 = 1$ in layers $z = 1, 3$ and $q_1 = -2X$ in the layer $z = 2$ by analogy with the $N_z = 2$ case) costs larger energy than in the case ii). The option i) is characterized by the vortex energy $E_v = \pi K_{11}(N_z/2)[(q_1 + 2Xq_2)^2 + (Y - 4X^2)q_2^2] \propto N_z$. The minimum for a $2X$ integer is achieved at $q_1 = -2Xq_2$, $q_2 = \pm 1$. Thus, in order to compensate for the factor $N_z \gg 1$, the system must be very close to the instability $0 < Y - 4X^2 < 2/N_z$. Simulations in this region turn out to be problematic. Thus, we will conduct simulations in the range $T_d < T < T_{(1,0)}$, provided the condition $T_d < T_{(1,0)}$ holds in the limit where $\delta = Y - 4X^2$ remains finite for $N_z \gg 1$. Specifically, the condition $T_d < \pi/2$ reads

$$\frac{1}{N_z} \sum_{m=0}^{N_z/2-1} \frac{1 + Y + 4X \cos^2 q_m}{Y - 4X^2 \cos^2 q_m} > 8. \quad (33)$$

It can surely be achieved for large enough X in the limit $N_z \gg 1$. Replacing the summation by integration in this limit and considering $\delta \ll 1$, Eq.(33) gives $\delta < (X/4\sqrt{2})^2$. For the simulations we have chosen $\delta = 0.3$ and $X = 6$, which gives $T_d \approx 0.983$ with $T = 1.28$ chosen in the middle of the interval between $T_{(1,0)} = \pi/2 \approx 1.57$ and T_d . The chosen value of T_d corresponds to the limit $N_z \rightarrow \infty$, and for any finite N_z , the actual T_d from Eq.(32) is below this value.

At this point we note that for any finite u the system is 3D and, strictly, speaking the notion of the BKT transition becomes inadequate for large enough N_z : even at $T > \pi/2$ the odd layers would still have XY order due to the proximity to even layers. Here, however, we presume that SP scenario holds and u_r vanishes at large L . Practically, simulations are performed at finite u and we always control that the XY stiffness (helicity modulus) along the layers remains finite and independent of L .

B. Dual formulation

The dual formulation in terms of the closed loops of integer J-currents (along bonds in and between the layers) can be achieved similarly to the case $N_z = 2$ by reinstating the compactness of ϕ_z in Eqs.(28,29) through the Villain approach: $\nabla_{ij}\phi_z \rightarrow (\nabla_{ij}\phi_z + 2\pi m_{z,ij})$ along the planes and $-u \cos(\phi_{z+1} - \phi_z) \rightarrow (u_V/2)(\phi_{z+1} - \phi_z + 2\pi m_{i,z})$ for Josephson coupling, where $m_{z,ij}$ refers to an arbitrary (oriented) integer defined on the bond ij belonging to the plane z and $m_{z,i}$ stands for an integer on a bond connecting site i in the plane z to the same site in the plane $z + 1$. The partition function is obtained as a result of integration over all $\phi_z(i)$ and summations over all bond integers.

The J-currents enter through the Poisson identity $\sum_{m=0,\pm 1,\pm 2,\dots} f(m) \equiv \int dx \exp(2\pi i J x) f(x)$ applied to each bond integer. This allows explicit integration over all phases ϕ_z as well as over the bond integers $m_{z,ij}, m_{i,z}$.

There are two types of J-currents: inplane $J_{z,ij}^{(a)}$, $a = 1, 2$ within each "elementary cell" (along z) and between the planes $J_{i,z}$. The label $a = 1$ refers to J-current defined on the bond ij belonging to a plane with odd z . Accordingly, $J_{z,ij}^{(2)}$ stands for the inplane current on the layer with even z . Then, $J_{i,z}$ denotes the current from the site i from the plane z to the plane $z + 1$. The integration over phases ϕ generates the Kirchhoff constraint — similarly to the bilayer case.

Finally the ensemble can be represented as

$$\begin{aligned} Z &= \sum_{\{\vec{J}\}} \exp(-H_J), \\ H_J &= \frac{1}{2} \sum_{ij;z,z'} V_{ab}(z - z') J_{z,ij}^{(a)} J_{z',ij'}^{(b)} \\ &\quad + \frac{1}{2u_V} \sum_{i,z} J_{i,z}^2, \end{aligned} \quad (34)$$

where the matrix $V_{ab}(z - z')$ is defined in terms of the matrix K_{ab} . It reflects the asymmetry between odd and even layers. Explicitly, $V_{11}(z) = Y V_{22}(z)$, for $z = z - z'$ being even, describes the interaction between odd layers, and $V_{22}(z)$ is defined between even layers; $V_{12}(z) = -X[V_{22}(z + 1) + V_{22}(z - 1)]$ refers to the interaction between odd and even layers (that is, z is odd), and

$$V_{22}(z) = \frac{2T}{N} \sum_{q_m} \frac{\cos(q_m z)}{Y - 4X^2 \cos^2(q_m)}, \quad (35)$$

with $z = 0, \pm 2, \pm 4, \dots$ and the summation running over $q_m = 4\pi m/N$, $m = 0, 1, \dots, N/2 - 1$.

The dual formulation for N_z layers allows obtaining the asymptotic expression for u_r within the same logic used for deriving Eq.(27). We will repeat it here. The loop formation can be viewed as a process of random walks of two ends of a broken loop — exactly along the line of the Worm Algorithm²⁰. Such a walk of each end is controlled by energetics of creating one bond element $|J| = 1$ in a randomly chosen direction — either along a given plane or perpendicular to it. Very similar to the case of the two layers, the energy to create such an element alone along the odd layer costs energy $\gg T \sim 1$, while the same element along an even layer costs energy ~ 1 . The only option for creating a loop in an odd layer is if its energy is compensated by parallel elements in the even plane. This feature is caused by the strong current-current interaction $\sim X$. Thus, if the walk occurs along z -direction from some even layer z toward the neighboring odd layer $z + 1$, the subsequent move along the odd layer will be too energetically costly so that the walker would either move further toward $z + 2$ layer or will go back to the original layer z . Thus, the inter-layer elements are characterized by either $J_{i,z} = J_{i,z+1} = \pm 1$ or $J_{i,z} = J_{i,z+1} = 0$. The weight of such a process is either $\exp(-1/u_V)$ or 1, respectively. Even if the walker makes a step or two along the layer $z + 1$ (which is a highly improbable event) and

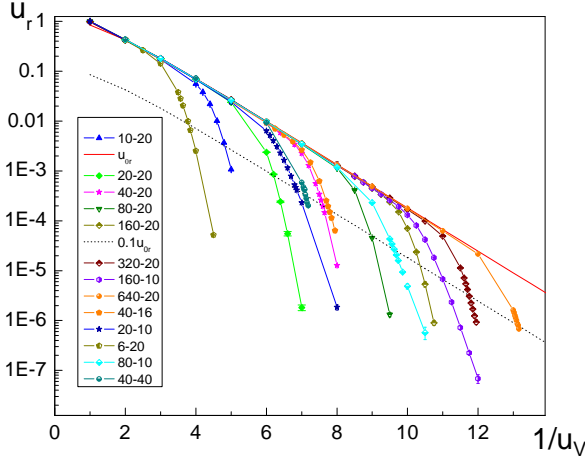


FIG. 3: (Color online) Monte Carlo results for the inter-layer stiffness u_r of the model (34,35) in the SP regime. Dashed orange line is the analytical solution (27). Dotted black line represents the offset $u_r = 0.1$ of the analytical solution (27). The first and the second numbers in the legend indicate values of L and N , respectively

then chooses to go toward the layer $z + 2$, the contribution to the partition function will be further reduced exponentially by the energy of the element J along the odd plane. Thus, such processes can be ignored, and we arrive at the conclusion that u_r given by Eq.(27) must be valid for arbitrary N_z in the asymptotic limit.

C. Numerical results

The model (34) has been simulated by the Worm Algorithm²⁰. The renormalized inter-layer stiffness u_r was found for a wide range of layer sizes, $6 \leq L \leq 640$ and $10 \leq N \leq 40$. The resulting data is presented in Figs. 3,4. As can be seen in Fig. 3, the solution (27) plays the role of the envelop for the family of the curves u_r vs $1/u_V$ for various L and N . We note that the stiffness ρ along the layers (as determined by Eq.(17)) remains independent of the sizes and much larger ($\rho = 22.6 \pm 0.5$) than u_r . This justifies the applicability of the asymptotic limit for Eq.(27). We have also controlled that the system is far enough from any possible composite phases¹⁷ state by measuring the lowest order correlator $\langle \exp(i\phi_z(x,y)) \exp(-i\phi_{z'}(x',y')) \rangle$ and observing that it exhibits long-range order for values $u \sim 1$ in the limit $N_z \sim L$. [In the composite phase state such a correlator is short ranged]. Thus, the system is well in the putative SP state. Its behavior, however, is drastically different from the RG prediction.

At this point we should discuss the deviations of the numerical curves from the analytical result seen in Fig.3. To some extent this behavior could have been interpreted as the evidence of SP. There is, however, one important observation: the value of $u_V = u^*$ below which such

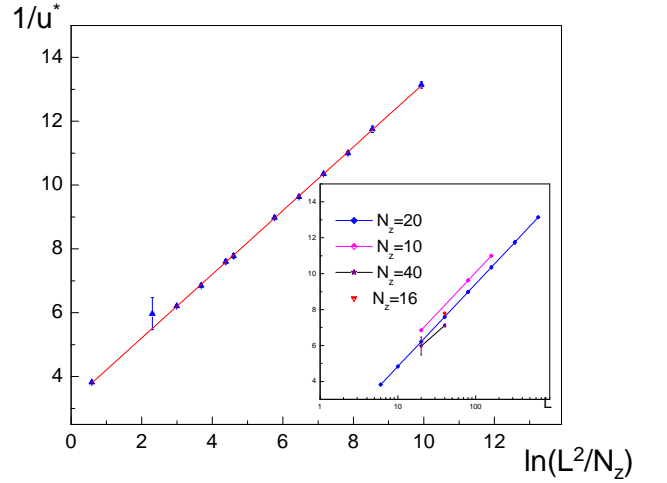


FIG. 4: (Color online) The values of u^* determined numerically from the data shown in Fig. 3 by finding the crossings of the curves u_r with the offset (dotted) line in Fig. 3. The linear fit of this line gives the slope $\gamma = 1.00 \pm 0.02$.

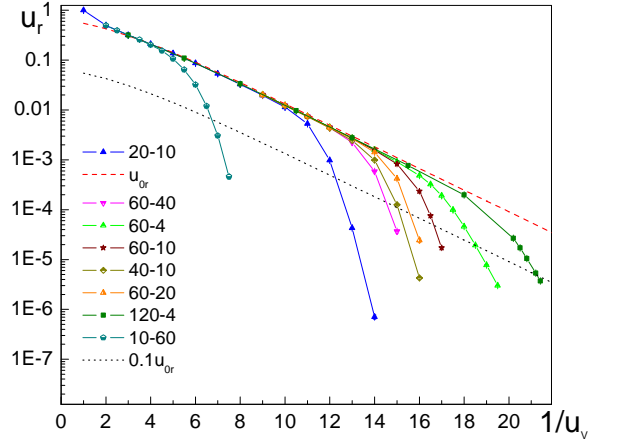


FIG. 5: (Color online) Monte Carlo results for the inter-layer stiffness u_r of the strongly asymmetric XY model. Dashed orange line is the analytical solution (25). Dotted black line represents the offset $u_r = 0.1$ of the analytical value (25). The first and the second numbers in the legend indicate values of L and N , respectively.

suppression begins decreases as

$$(u^*)^{-1} = \gamma \ln(L^2/N_z), \quad \gamma = 1.00 \pm 0.02 \quad (36)$$

for $L^2/N_z \gg 1$ in the main logarithmic approximation. This behavior is demonstrated in Fig.4, where the value u^* corresponds the offset for u_r being taken at $1/10$ of the value given by the analytical expression (27). The suppression of $u^* \rightarrow 0$ in the limit $L \rightarrow \infty$ indicates that it is a finite size effect. In other words, starting from small L at some $u_V < 1$, the renormalized stiffness u_r can be well below the asymptotic limit (27). As L increases while u_V is kept fixed, u_r eventually approaches the limit (27). This feature is clearly in a stark contrast with the RG prediction of SP where u_r is supposed to flow to zero for fixed u_V as a power of $L \rightarrow \infty$.

The deviation from the limit (27) has a very simple

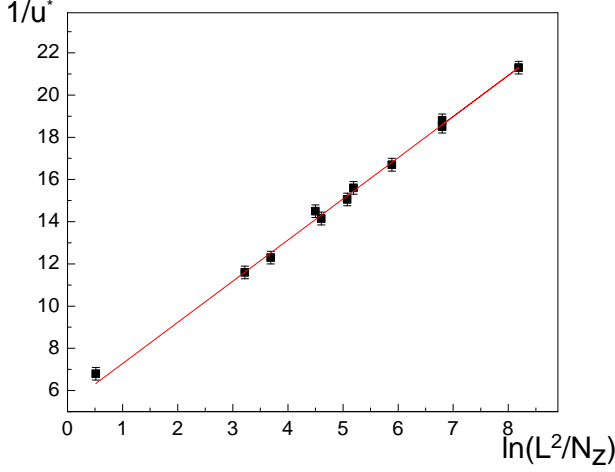


FIG. 6: (Color online) The values of u^* determined numerically from the data shown in Fig. 5 by finding the crossings of the curves u_r with the offset (dotted) line in Fig. 5. The linear fit of this line gives the slope 1.95 ± 0.05 .

explanation: it is essentially a consequence of the generic exponential suppression of any order in a quasi-1D limit $N_z \rightarrow \infty$. [There is no such suppression in the case of $N_z = 2$ because the loops along z -direction are independent from the inplane ones]. In the context of our system it can be interpreted as the effect of zero modes (in each layer) fluctuations. Indeed, excitations along the planes renormalize u_V to u_r at short distances. Then, as long as $u_r L^2 \ll K_{11}$ the only remaining lowest energy degrees of freedoms are zero modes, that is, excitations with $\phi_z = \phi_z^{(0)}$ being independent from the x, y positions along the planes (and dependent on z). Then, the effective action becomes

$$S_0 = -u_r L^2 \sum_z [\cos(\phi_{z+1}^{(0)} - \phi_z^{(0)})]. \quad (37)$$

It should be used in the partition function $Z = \int D\phi_z^{(0)} \exp(-S_0)$. Its analytical evaluation gives

$$u_r^{(0)} \sim \exp(-N_z/(u_r L^2)) \quad (38)$$

for the value of the stiffness along z -direction – that is, the renormalised u_r on large scale. Keeping in mind that at short scales $u_r \sim \exp(-1/u_V)$ for $u_V \ll 1$, one arrives at the relation (36) with $\gamma = 1$. It corresponds to the requirement $N_z/(u_r L^2) \approx 1$ for some $u_V = u^*$ so that Eq.(38) describes the exponential suppression at $u_V < u^*$.

Clearly, such a quasi-1D suppression is also present in the standard XY model (where no SP are anticipated to exist). In order to demonstrate this explicitly we have also simulated a simple XY model given by the system

$$Z_{XY} = \int D\phi_z \exp(-H_{XY}), \quad (39)$$

$$H_{XY} = - \sum_{\langle ij \rangle, z} [\tilde{K} \cos(\nabla_{ij} \phi_z) + u \cos(\nabla_z \phi_z)],$$

with some $\tilde{K} \gg 1$ (guaranteeing that no BKT transition occurs in each layer for $u = 0$), and $0 < u < \tilde{K}$. In the dual representation this system is described by

$$H_{XY} \rightarrow \tilde{H}_{XY} = \sum_{\langle ij \rangle, z} \frac{1}{2\tilde{K}} J_{z,ij}^2 + \sum_{i,z} \frac{1}{2u_V} J_{i,z}^2, \quad (40)$$

where $J_{ij,z}$ and $J_{i,z}$ are the same J-currents introduced above for the model (34). The results of the simulations of this model are presented in Fig.5,6. In the asymptotic limit the inter-layer stiffness is described by Eq.(25). Then, according to the above discussion the value u^* determining the start of the deviations is given by $(u^*)^{-1} = 2 \ln(L^2/N_z)$, that is, with the slope $\gamma = 2$ which should be compared with the numerical value $\gamma = 1.95 \pm 0.05$ in Fig.6.

Thus, both models demonstrate essentially the same 3D behavior. The only difference is the slope of the renormalized Josephson coupling $\ln u_r$ vs its bare value u_V . It is determined by the minimal length of the elementary J-current in the direction perpendicular to the layers.

IV. DISCUSSION

The RG approach to 2D systems proves to be every effective in many cases including 2D XY model when it can be mapped on the Sine-Gordon (SG) one²⁵. A successful implementation of the RG analysis to the Josephson coupling was demonstrated in Ref.²⁶ where a single weak link can make one channel Luttinger liquid insulating.

The merit of RG, however, should be taken with caution when applied to the dimensional reduction situations in layered systems. In this case there is no exact mapping between XY and SG representations at finite inter-layer Josephson coupling, and the approximation ignoring the compact nature of the variables becomes uncontrolled. As our analysis of one particular layered system shows, no SP exists in such a system despite the RG prediction: the system shows essentially the 3D behavior of the asymmetric XY model.

While it is not obvious to us what is wrong with treating Josephson coupling between 2D layers by RG in the original representation of fields, the dual formulation in terms of the closed loops gives a very important insight. Specifically, the SP means that as layer size $L \rightarrow \infty$, a suppression of the Josephson coupling between layers would require that the number of times elements of closed loops fluctuate between layers must scale slower than L^2 so that the density of such events is zero in the limit $L = \infty$. The loops statistics, however, is controlled by local energies of creating finite elements and the entropy due to 6 directions in 3D vs 4 along layers. Thus, as long as there is a finite energy to cross between neighboring layers, the entropy will lead to a finite density of crossings for large enough L . Similar argument can be applied to quantum wires in terms of the quantum to classical mapping where imaginary time is treated as an extra dimension.

The dual approach and the argumentation along the line of the numerical algorithm²⁰, treating closed loops formation as a process of the worm head wondering around and eventually finding its tail, allowed us to expose the actual stages of the renormalization of the Josephson coupling: i) At small scales Josephson coupling is controlled by exponentially suppressed random and independent (in the asymptotic limit) events of crossings between layers. It can be viewed as an ideal gas of J-currents between the layers. This stage leads to the renormalized coupling, in general, represented by Eq.(24) with $M = 1, 2, 3, \dots$ ii) If the number of layers N_z increases, with L being fixed, quasi 1D fluctuations further suppress the coupling exponentially as demonstrated in Eq.(38).

Here we have discussed a local model characterized by short range interactions between the inter-layer J-current elements. This feature in combination with the low density of such elements justifies the "ideal gas" approximation for them, which in its turn leads to finite values of the renormalized inter-layer Josephson coupling.

The question may be raised if a presence of long-range forces between the inter-plane J-currents J_i can change the situation and lead to the SP or its weaker version – where $u_r \rightarrow 0$ with the growing number of layers N_z in the limit $L = \infty$. In this respect we note that in order to realize this, fluctuations of the difference of the J-currents with positive and negative orientations must be macroscopically suppressed. In this case the fluctuation of the winding numbers in z -direction $\langle W_z^2 \rangle$ will scale slower than L^2 so that $u_r \sim \langle W_z^2 \rangle / L^2 \rightarrow 0$. This may be caused by interactions between the inter-layer J-currents decaying not faster than the second power of their separation. More specifically, the following additional repulsive term

$$H_{SP} = \frac{1}{2} \sum_{i,j} U(\vec{x}_i - \vec{x}_j) J_i J_j \quad (41)$$

in the simple XY J-current model (40) with $U(\vec{x})$ having the long range tail $\sim 1/|\vec{x}|^\sigma$ with $\sigma < 2$ will generate the energy contribution $\sim W_z^2 L^{-\sigma}$ in terms of the windings in z -direction. Consequently, the renormalized Josephson coupling (19) would scale as $u_r \sim L^{\sigma-2} \rightarrow 0$.

In one particular example long-range forces are introduced into the inter-layer Josephson in the standard XY model (39) by some effective gauge-type term $-u \cos(\nabla_z \phi - g_z A_z) + (\vec{\nabla} A_z)^2$, where $\vec{\nabla} A_z$ refers to the derivatives along the layers of some soft mode A_z , with g_z being a parameter. The resulting interaction in the dual form (41) becomes $U \sim g_z^2 \ln(|\vec{x}|)$ and, thus, it suppresses the inter-layer Josephson as $u_r = N_z \langle W_z^2 \rangle / L^2 \sim 1/(L^2 \ln L)$ in the limit $L \rightarrow \infty$ for fixed N_z .

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